

BUCHSBAUM-RIM MULTIPLICITIES AS HILBERT-SAMUEL MULTIPLICITIES

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ABSTRACT. We study the Buchsbaum-Rim multiplicity $\text{br}(M)$ of a finitely generated module M over a regular local ring R of dimension 2 with maximal ideal \mathfrak{m} . The module M under consideration is of finite colength in a free module F . Write $F/M \cong I/J$, where $J \subset I$ are \mathfrak{m} -primary ideals of R . We first investigate the colength $\ell(R/\mathfrak{a})$ of any \mathfrak{m} -primary ideal \mathfrak{a} and its Hilbert-Samuel multiplicity $e(\mathfrak{a})$ using linkage theory. As applications, we establish several multiplicity formulas that express the Buchsbaum-Rim multiplicity of the module M in terms of the Hilbert-Samuel multiplicities of ideals related to I , J and an arbitrary general minimal reduction of M . The motivation comes from E. Jones' article [12] who applied graphical computations of the Hilbert-Samuel multiplicity to the Buchsbaum-Rim multiplicity.

Let R be a local Cohen-Macaulay ring with maximal ideal \mathfrak{m} and infinite residue field. Let \mathfrak{a} be an \mathfrak{m} -primary ideal. In this paper, we study the connection between the *colength* of \mathfrak{a} , *i.e.*, the length $\ell(R/\mathfrak{a})$ of R/\mathfrak{a} , and the Hilbert-Samuel multiplicity $e(\mathfrak{a})$ of \mathfrak{a} . It is known for an \mathfrak{m} -primary ideal \mathfrak{b} contained in \mathfrak{a} that $e(\mathfrak{a}) = e(\mathfrak{b})$ if and only if \mathfrak{b} is a reduction of \mathfrak{a} . Furthermore, if \mathfrak{b} is a minimal reduction of \mathfrak{a} , then

$$(1) \quad e(\mathfrak{a}) = e(\mathfrak{b}) = \ell(R/\mathfrak{b}).$$

However, $e(\mathfrak{a})$ and $\ell(R/\mathfrak{a})$ are not equal in general. In one of our main theorems, Theorem 2.2, we express, under certain conditions, the colength of \mathfrak{a} in terms of the Hilbert-Samuel multiplicity of ideals which are in the same linkage class of \mathfrak{a} .

Equation (1) can be generalized to modules using the Buchsbaum-Rim multiplicity of a module M , denoted $\text{br}(M)$. Let $N \subset M$ be submodules contained in a free module F of finite rank such that $\ell(F/N) < \infty$. It is known that N and M have the same Buchsbaum-Rim multiplicity if and only if N is a reduction of M . Similar to ideals, if N is a minimal reduction of M , then

$$(2) \quad \text{br}(M) = \text{br}(N) = \ell(F/N)$$

(*cf.* [5], [13], [18]).

In the case where F has rank one, M is an \mathfrak{m} -primary ideal and $\text{br}(M) = e(M)$. We see that the Buchsbaum-Rim multiplicity is a generalization of the Hilbert-Samuel multiplicity to modules. Like the Hilbert-Samuel multiplicity, it characterizes reductions. Using the theory of reductions of modules, we reduce the problem of finding formulas for the Buchsbaum-Rim multiplicity to the relationship between the colength

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and the Hilbert-Samuel multiplicity of ideals. The latter question is answered for arbitrary licci ideals in Theorem 2.2. As an application, we obtain formulas for the Buchsbaum-Rim multiplicity of a two-dimensional module in terms of the Hilbert-Samuel multiplicities of a certain Fitting ideal and ideals linked to it, see Theorem 2.4. We also prove expressions for the Buchsbaum-Rim multiplicity that involve Bourbaki ideals associated to the module, see Theorems 3.1, 3.3, and Corollary 3.4. The last corollary contains the work of [12] as a special case.

The paper is arranged in the following way: Section 1 introduces the notion of the Buchsbaum-Rim multiplicity and its basic properties. We also include the definitions of some notation and theorems that will be used in the later sections. In Section 2, we state and prove the main theorem that relates the colength and the Hilbert-Samuel multiplicity of \mathfrak{m} -primary ideals in regular local rings of dimension two. In Section 3, we discuss several multiplicity formulas that express the Buchsbaum-Rim multiplicity of a module in terms of the Hilbert-Samuel multiplicity of \mathfrak{m} -primary ideals related to the module. Jones [12] provides a method for computing the Buchsbaum-Rim multiplicity of modules of a special type. In Section 4 we compare the multiplicity formulas obtained in Section 3 to the results of [12].

1. INTRODUCTION TO THE BUCHSBAUM-RIM MULTIPLICITY

In 1964, Buchsbaum and Rim [5] introduced and studied the multiplicity that bears their names. It was further studied by Gaffney, Kirby, Rees and many others, including Kleiman and Thorup who investigated the geometric theory of the Buchsbaum-Rim multiplicity in [13]. In this paper, we study the connection between the Buchsbaum-Rim multiplicity and the Hilbert-Samuel multiplicity.

Throughout the paper, we assume that R is a Noetherian local ring of dimension d with maximal ideal \mathfrak{m} . Let \mathfrak{a} be an \mathfrak{m} -primary ideal of R . There exists a polynomial $P_{\mathfrak{a}}(n)$ of degree d such that $P_{\mathfrak{a}}(n) = \ell(R/\mathfrak{a}^n)$ for large $n \in \mathbb{N}$. This polynomial is called the Hilbert-Samuel polynomial and the coefficient of $\frac{n^d}{d!}$ is the *Hilbert-Samuel multiplicity* $e(\mathfrak{a})$.

The Buchsbaum-Rim multiplicity can be viewed as a generalization of the Hilbert-Samuel multiplicity. For a submodule M of finite colength in a free module F of rank r , Buchsbaum and Rim [5] prove that there exists a polynomial $\lambda(n)$ of degree $d + r - 1$ such that for all large $n \in \mathbb{N}$,

$$\lambda(n) = \ell(\mathcal{S}_n(F)/\mathcal{R}_n(M)),$$

where $\mathcal{S}(F) = \bigoplus_{n \geq 0} \mathcal{S}_n(F)$ is the symmetric algebra of F and $\mathcal{R}(M) = \bigoplus_{n \geq 0} \mathcal{R}_n(M)$ is the image of the natural map $\mathcal{S}(M) \rightarrow \mathcal{S}(F)$. Notice that the algebra $\mathcal{R}(M)$ is the R -subalgebra of $\mathcal{S}(F)$ generated by M . The *Buchsbaum-Rim multiplicity* $\text{br}(M)$ is defined to be the coefficient of $\frac{n^{d+r-1}}{(d+r-1)!}$ in the polynomial $\lambda(n)$. Buchsbaum and Rim showed that $\text{br}(M)$ is a positive integer if M is a proper submodule of F . Notice that if $r = 1$, then M is an \mathfrak{m} -primary ideal of R , $\lambda(n) = P_M(n)$ and $\text{br}(M) = e(M)$.

If $\text{depth } R \geq 2$, then the inclusion $M \subset F$ where $\ell(F/M) < \infty$ can be identified with the natural embedding of M into its double dual M^{**} . Hence in this case $\text{br}(M)$

is independent of the embedding of M into a free module. Moreover, if R is a two-dimensional regular local ring, one can define the Buchsbaum-Rim multiplicity of *any* finitely generated R -module M : simply consider the natural map from M to M^{**} , which is free in this case, and replace M by its image under this map.

Let F be a free R -module of rank r , let M be a submodule of F with $\ell(F/M) < \infty$, and let U be a submodule of M . Again, we write $\mathcal{R}(U)$ and $\mathcal{R}(M)$ for the R -subalgebras of $\mathcal{S}(F)$ generated by U and M , respectively. We say that U is a *reduction* of M if $\mathcal{R}(M)$ is integral over $\mathcal{R}(U)$ as rings. A minimal reduction of M is a reduction that is minimal with respect to inclusion. Notice that a reduction U of M is minimal if and only if its minimal number of generators is $r + d - 1$, at least when the residue field of R is infinite [16].

On the other hand, if we fix a basis for F , then the submodule M of F is associated with a matrix, denoted by \widetilde{M} , whose columns are the generators of M . Recall that the zeroth Fitting ideal $\text{Fitt}_0(F/M)$ is the ideal generated by the maximal minors of \widetilde{M} . This ideal is independent of the choices of the generators of M and the basis of F . We recall a theorem by Rees relating reductions of ideals and modules:

Theorem 1.1. (Rees[16, 1.2]) The submodule U of M is a reduction of M if and only if the subideal $\text{Fitt}_0(F/U)$ is a reduction of $\text{Fitt}_0(F/M)$.

If U is a reduction of M , then $\text{br}(U) = \text{br}(M)$. The converse holds in the case R is equidimensional and universal catenary (cf. [13], [18]).

The following theorem relates the notions of reductions and Buchsbaum-Rim multiplicity.

Theorem 1.2. (Buchsbaum-Rim[5, 4.5(2)], Kleiman-Thorup[13, 5.3(i)], Simis-Ulrich-Vasconcelos[18, 2.5], Bruns-Vetter[4, 2.8 and 2.10]) Assume that R is a Cohen-Macaulay local ring with infinite residue field. If U is a minimal reduction of M , then

$$\text{br}(M) = \text{br}(U) = \ell(F/U) = \ell(R/\text{Fitt}_0(F/U)).$$

Let R be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field and let M be a finitely generated module of finite colength in a free module F of rank r . For such a module M , there exist ideals $J \subset I$ of height ≥ 2 such that F/M is isomorphic to I/J . In fact one can take $I \cong F/G$ to be a Bourbaki ideal of F with $G \subset M$ a free submodule of rank $r - 1$ and J to be the image of M in I (cf. [3, Chapter 7 no. 4, Theorem 6], [19, 3.2(a)(c)]). Notice that if $d = 2$ and $\mu(M) \leq 3$ then I and J can be chosen to be complete intersections. Since M is its own minimal reduction in this case, we obtain the following equalities by Theorem 1.2,

$$(3) \quad \text{br}(M) = \ell(F/M) = \ell(R/J) - \ell(R/I) = e(J) - e(I).$$

We see that the Buchsbaum-Rim multiplicity is connected to the Hilbert-Samuel multiplicity in this special case (cf. [12]). We are interested in such a relationship for arbitrary modules. By Theorem 1.2, $\text{br}(M)$ is equal to the colength of the Fitting ideal corresponding to a minimal reduction of M . Thus, the question can be reduced to

investigating the connection between the colength and the Hilbert-Samuel multiplicity of ideals.

2. COLENGTH AND THE HILBERT-SAMUEL MULTIPLICITY

In a Cohen-Macaulay local ring R , two proper ideals \mathfrak{a} and \mathfrak{a}_1 are *linked* with respect to a complete intersection ideal \mathfrak{c} , denoted $\mathfrak{a} \sim \mathfrak{a}_1$, if $\mathfrak{a}_1 = \mathfrak{c} : \mathfrak{a}$ and $\mathfrak{a} = \mathfrak{c} : \mathfrak{a}_1$. If R is local Gorenstein and \mathfrak{a} is unmixed of grade g (i.e., $\dim R_{\mathfrak{p}} = g$ for all associated prime ideals \mathfrak{p} of R/\mathfrak{a}), it suffices to require $\mathfrak{a}_1 = \mathfrak{c} : \mathfrak{a}$. We say an ideal \mathfrak{a} is *in the linkage class of a complete intersection* (or \mathfrak{a} is *licci* for simplicity) if there are ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ with $\mathfrak{a} \sim \mathfrak{a}_1 \sim \dots \sim \mathfrak{a}_n$ and \mathfrak{a}_n a complete intersection.

Theorem 2.1. (Huneke-Ulrich [11, proof of 2.5]) Let (R, \mathfrak{m}) be a Gorenstein local ring with infinite residue field and let \mathfrak{a} be a licci \mathfrak{m} -primary ideal linked to a complete intersection in n steps. Then there exists a sequence of links $\mathfrak{a} = \mathfrak{a}_0 \sim \mathfrak{a}_1 \sim \dots \sim \mathfrak{a}_n$ such that \mathfrak{a}_n is a complete intersection, and \mathfrak{a}_i and \mathfrak{a}_{i+1} are linked with respect to a minimal reduction of \mathfrak{a}_i .

Theorem 2.2. *In the setting of Theorem 2.1, we have*

$$\ell(R/\mathfrak{a}) = \sum_{i=0}^n (-1)^i e(\mathfrak{a}_i).$$

Proof. If \mathfrak{a} is a complete intersection, then $\ell(R/\mathfrak{a}) = e(\mathfrak{a})$ and the assertion is clear. We assume that \mathfrak{a} is not a complete intersection and do induction on n . Let \mathfrak{b}_0 be a minimal reduction of \mathfrak{a} such that $\mathfrak{a}_1 = \mathfrak{b}_0 : \mathfrak{a}$. Notice that $e(\mathfrak{b}_0) = e(\mathfrak{a})$. The quotient ring R/\mathfrak{b}_0 is Gorenstein since \mathfrak{b}_0 is generated by a regular sequence. Moreover,

$$(\mathfrak{b}_0 : \mathfrak{a})/\mathfrak{b}_0 = \operatorname{Hom}_{R/\mathfrak{b}_0}(R/\mathfrak{a}, R/\mathfrak{b}_0) = \operatorname{Hom}_{R/\mathfrak{b}_0}(R/\mathfrak{a}, \omega_{R/\mathfrak{b}_0}) = D_{R/\mathfrak{b}_0}(R/\mathfrak{a}),$$

where $\omega_{R/\mathfrak{b}_0}$ is the canonical module of R/\mathfrak{b}_0 and D denotes the dualizing functor. Since the dualizing functor preserves length, we have

$$\ell((\mathfrak{b}_0 : \mathfrak{a})/\mathfrak{b}_0) = \ell(R/\mathfrak{a}).$$

Therefore

$$\begin{aligned} \ell(R/\mathfrak{a}) &= \ell(R/\mathfrak{b}_0) - \ell(R/(\mathfrak{b}_0 : \mathfrak{a})) \\ &= e(\mathfrak{b}_0) - \ell(R/\mathfrak{a}_1) \\ &= e(\mathfrak{a}) - \ell(R/\mathfrak{a}_1). \end{aligned}$$

The result now follows by induction. □

Corollary 2.3. *Let (R, \mathfrak{m}) be a regular local ring of dimension 2 with infinite residue field. If \mathfrak{a} is an integrally closed \mathfrak{m} -primary ideal, then*

$$\ell(R/\mathfrak{a}) = \sum_{i=1}^{\infty} (-1)^{i+1} e(\operatorname{Fitt}_i(\mathfrak{a})).$$

Proof. We induct on $\mu(\mathfrak{a})$, the case $\mu(\mathfrak{a}) = 2$ being obvious. Thus let $\mu(\mathfrak{a}) = r + 1 \geq 3$. Choose $\mathfrak{a}_0, \dots, \mathfrak{a}_r$ as in Theorem 2.2, and notice that $\mu(\mathfrak{a}_1) < \mu(\mathfrak{a})$. By Huneke and Swanson [10, 3.4 and 3.5], \mathfrak{a}_1 is integrally closed, $\mathfrak{a}_1 = \operatorname{Fitt}_2(\mathfrak{a})$, and

$\text{Fitt}_i(\mathfrak{a}_1) = \text{Fitt}_{i+1}(\mathfrak{a})$ for every $i \geq 1$. Now we apply Theorem 2.2 and the induction hypothesis. \square

In Theorem 2.2 the colength of a licci ideal is expressed in terms of Hilbert-Samuel multiplicities. This result applies to any \mathfrak{m} -primary perfect ideal in a two-dimensional Gorenstein local ring with infinite residue field. For three-dimensional rings, J. Watanabe [20] has proved that every \mathfrak{m} -primary perfect Gorenstein ideal is licci.

Theorem 2.4. *Let R be a Gorenstein local ring of dimension 2 with infinite residue field, let M be a submodule of finite colength in a free module F of rank r , and let U be a minimal reduction of M .*

- (a) *There exists a sequence of links $\text{Fitt}_0(F/U) = \mathfrak{a}_0 \sim \mathfrak{a}_1 \sim \cdots \sim \mathfrak{a}_{r-1}$ such that \mathfrak{a}_{r-1} is a complete intersection, and \mathfrak{a}_i and \mathfrak{a}_{i+1} are linked with respect to a minimal reduction of \mathfrak{a}_i .*
- (b)

$$\text{br}(M) = e(\text{Fitt}_0(F/M)) + \sum_{i=1}^{r-1} (-1)^i e(\mathfrak{a}_i).$$

Proof. To prove part (a), notice that $\mathfrak{a} = \text{Fitt}_0(F/U)$ has height 2 and is generated by the maximal minors of an r by $r+1$ matrix. Thus \mathfrak{a} can be linked to a complete intersection in $r-1$ steps $\mathfrak{a} = \mathfrak{a}_0 \sim \mathfrak{a}_1 \sim \cdots \sim \mathfrak{a}_{r-1}$ (cf. [1], [2], [9]). By Theorem 2.1 we may assume that \mathfrak{a}_i and \mathfrak{a}_{i+1} are linked with respect to a minimal reduction of \mathfrak{a}_i .

Part (b) follows from (a), Theorems 1.1, 1.2 and 2.2. \square

Remark 2.5. The ideals \mathfrak{a}_i , $0 \leq i \leq r-1$, of Theorem 2.4 can be obtained concretely in the following way: After applying general row and column operations to the matrix \widetilde{M} presenting F/M , the ideal \mathfrak{a}_i is generated by the maximal minors of the matrix consisting of the last $r - 2\lceil \frac{i-1}{2} \rceil$ rows and the last $r+1 - 2\lceil \frac{i}{2} \rceil$ columns of \widetilde{M} ([2]).

As an immediate consequence of Corollary 2.3 and Theorem 2.4, we obtain that if $\text{Fitt}_0(F/U)$ is integrally closed, then

$$\text{br}(M) = \sum_{i=0}^{r-1} (-1)^i e(\text{Fitt}_i(F/U)).$$

The following remark provides another point of view on the formula in Theorem 2.4.

Remark 2.6. As in Remark 2.5 we apply general row and column operations to the matrix \widetilde{M} , and then obtain an exact sequence

$$R^n \xrightarrow{\widetilde{M}} F \longrightarrow C_0 = F/M \longrightarrow 0.$$

The Auslander dual $D(C_0)$ of C_0 is presented by \widetilde{M}^* ,

$$F^* \xrightarrow{\widetilde{M}^*} R^{n^*} \longrightarrow D(C_0) \longrightarrow 0.$$

Let C_1 be the quotient of $D(C_0)$ modulo the submodule generated by the image of the last $n - r + 1$ basis elements of R^n . The submatrix of \widehat{M}^* involving the top $r - 1$ rows presents C_1 .

Continuing this way, we obtain a sequence of modules C_0, \dots, C_{r-1} , where C_i is the quotient of $D(C_{i-1})$ modulo the submodule generated by the last two generators. Notice that C_i is represented by the transpose of the matrix consisting of the first $r - 2\lceil \frac{i-1}{2} \rceil$ rows and the first $r + 1 - 2\lceil \frac{i}{2} \rceil$ columns of \widehat{M} described in Remark 2.5. Hence $\text{Fitt}_0(C_i) = \mathfrak{a}_i$ and by Theorem 2.4,

$$\text{br}(M) = \sum_{i=0}^{r-1} (-1)^i e(\text{Fitt}_0(C_i)).$$

3. MULTIPLICITY FORMULAS

In this section, we discuss other connections between the Buchsbaum-Rim multiplicity of modules and the Hilbert-Samuel multiplicity of ideals. In fact, we relate the Buchsbaum-Rim multiplicity of M to the Hilbert-Samuel multiplicity of a generic Bourbaki ideal of F with respect to M , see Theorem 3.1. However, if there is a need to fix a certain Bourbaki ideal I of F , the result in Theorem 3.1 does not apply anymore. Instead Theorem 3.3 takes care of these cases.

Theorem 3.1. *Let (R, \mathfrak{m}) be a Gorenstein local ring of dimension 2 with infinite residue field, let M be a submodule of finite colength in a free module F of rank r , let U be a minimal reduction of M , and let \mathfrak{a}_i be ideals as in Theorem 2.4(a). Then there exists an \mathfrak{m} -primary Bourbaki ideal I of F and a subideal $J \subset I$ such that $F/M \cong I/J$ and*

$$\text{br}(M) = e(J) - e(I) + e(\mathfrak{a}_2) + \dots + (-1)^{r-1} e(\mathfrak{a}_{r-1}).$$

In particular, if $\text{rank } M = 2$, then there exist \mathfrak{m} -primary ideals $J \subset I$ such that $F/M \cong I/J$ and

$$\text{br}(M) = e(J) - e(I).$$

Proof. We may assume $r \geq 2$. Let \mathfrak{b}_0 be a minimal reduction of $\mathfrak{a}_0 = \text{Fitt}_0(F/U)$ defining the link $\mathfrak{a}_0 \sim \mathfrak{a}_1$. We can find generators u_1, \dots, u_{r+1} of U in F so that \mathfrak{a}_0 and \mathfrak{a}_1 are the ideals of maximal minors of the matrices $\widetilde{U} = (u_1 | \dots | u_{r+1})$ and $\widetilde{V} = (u_1 | \dots | u_{r-1})$, and \mathfrak{b}_0 is generated by the determinants of $(u_1 | \dots | u_{r-1} | u_{r+1})$ and $(u_1 | \dots | u_{r-1} | u_r)$.

Let G be the submodule of U generated by u_1, \dots, u_{r-1} . As $\mathfrak{a}_1 = I_{r-1}(\widetilde{V})$ has height 2, it follows that G is free and $\mathfrak{a}_1 \cong F/G$ is an \mathfrak{m} -primary Bourbaki ideal of F . Thus we may take I to be \mathfrak{a}_1 .

Now let J be the image of M in I . Clearly $J \cong M/G$ and hence $I/J \cong F/M$. Notice that \mathfrak{b}_0 is the image of U in I . As U is a reduction of M , it follows that \mathfrak{b}_0 is a reduction of J . Since by definition \mathfrak{b}_0 is also a reduction of \mathfrak{a}_0 , we deduce $e(J) = e(\mathfrak{b}_0) = e(\mathfrak{a}_0)$. Now Theorem 2.4 gives

$$\text{br}(M) = e(J) - e(I) + e(\mathfrak{a}_2) + \dots + (-1)^{r-1} e(\mathfrak{a}_{r-1}). \quad \square$$

We would like to point out that the result in Theorem 3.1 does not hold for an arbitrary pair of Bourbaki ideals $J \subset I$ satisfying $F/M \cong I/J$. This case is treated in our next result. Theorem 3.3 provides an expression for $\text{br}(M)$ in terms of $e(I)$ and $e(J)$ if I and J are already specified. This is motivated by the work in Jones [12] where it is necessary to choose I and J to be monomial ideals in order to extend the graphical computation of the Hilbert-Samuel multiplicity of monomial ideals to the Buchsbaum-Rim multiplicity of modules. Jones also provides a class of examples where the formula of Theorem 3.1 does not hold for arbitrary Bourbaki ideals $J \subset I$.

Assumption 3.2. Let (R, \mathfrak{m}) be a Gorenstein local ring of dimension 2 with infinite residue field, let M be a submodule of finite colength in a free module F of rank r , and assume M has no free direct summand. Write $F/M \cong I/J$, where $J \subset I$ are \mathfrak{m} -primary ideals, I has finite projective dimension, and $\mu(I) \leq r$. Since $M \subset \mathfrak{m}F$, we have $\mu(I/J) = \mu(F/M) = \mu(F) = r \geq \mu(I)$ and therefore $J \subset \mathfrak{m}I$. Thus the lift $F \rightarrow I$ of the above isomorphism is surjective by Nakayama's Lemma. It induces an isomorphism $I \cong F/G$, where G is a free submodule of F of rank $r - 1$. By restriction we obtain $J \cong M/G$.

Let s_1, \dots, s_{r-1} be generators of G and let z_r, \dots, z_{2r} be generators of a minimal reduction U of M . Thinking of $s_i \in F$ and $z_j \in F$ as column vectors we form the matrices

$$\tilde{L} = (s_1 | \dots | s_{r-1} | z_r | \dots | z_{2r}), \quad \tilde{U} = (z_r | \dots | z_{2r}), \quad \tilde{N} = (s_1 | \dots | s_{r-1} | z_{2r-1} | z_{2r}).$$

By performing row operations on \tilde{L} and by adding suitable linear combinations of columns of \tilde{L} to later columns we may achieve these properties:

- s_1, \dots, s_{r-1} still generate G .
- z_r, \dots, z_{2r} still generate a minimal reduction U of M .
- the images of z_{2r-1}, z_{2r} in $M/G = J$ generate a minimal reduction J' of J .
- if for each i with $0 \leq i \leq r - 1$, J_i denotes the ideal of maximal minors of the matrix consisting of the last $r - 2\lceil \frac{i-1}{2} \rceil$ rows and the last $r + 1 - 2\lceil \frac{i}{2} \rceil$ columns of \tilde{U} , then J_i and J_{i+1} are linked with respect to a minimal reduction of J_i for $0 \leq i \leq r - 2$.
- if for each i with $0 \leq i \leq r - 1$, J'_i denotes the ideal of maximal minors of the matrix consisting of the last $r - 2\lceil \frac{i-1}{2} \rceil$ rows and the last $r + 1 - 2\lceil \frac{i}{2} \rceil$ columns of \tilde{N} , then J'_i and J'_{i+1} are linked with respect to a minimal reduction of J'_i for $0 \leq i \leq r - 2$. Notice that $J_{r-1} = J'_{r-1}$ and if r is odd then also $J_{r-2} = J'_{r-2}$.

Finally, let I' be any minimal reduction of I , and $\text{Fitt}_0(I/I') = I_0 \sim I_1 \sim \dots \sim I_{r-3}$ be a sequence of links as in Theorem 2.1.

Note that for the last two conditions in 3.2, one only has to check that the two minors corresponding to the first two rows or columns in the matrix of J_i (or J'_i) generate a reduction of J_i (resp. J'_i).

Theorem 3.3. *With assumptions as in 3.2 one has*

$$\begin{aligned} \text{br}(M) = e(J) - e(I) &+ (e(\text{Fitt}_0(I/J)) + E_U) - (e(\text{Fitt}_0(I/J')) + E_N) \\ &+ (e(\text{Fitt}_0(I/I')) + E_I), \end{aligned}$$

where $E_U = \sum_{i=1}^{2\lfloor \frac{r-2}{2} \rfloor} (-1)^i e(J_i)$, $E_N = \sum_{i=1}^{2\lfloor \frac{r-2}{2} \rfloor} (-1)^i e(J'_i)$, and $E_I = \sum_{i=1}^{r-3} (-1)^i e(I_i)$.

Proof. As U is a reduction of M , Theorem 1.1 shows that $I_r(\tilde{U})$ is a reduction of $\text{Fitt}_0(F/M) = \text{Fitt}_0(I/J)$. Therefore

$$e(I_r(\tilde{U})) = e(\text{Fitt}_0(I/J)).$$

Applying Theorem 2.2 to the ideals $I_r(\tilde{U})$, $I_r(\tilde{N}) = \text{Fitt}_0(I/J')$ and $\text{Fitt}_0(I/I')$ we obtain

$$(4) \quad \ell(R/I_r(\tilde{U})) = \begin{cases} e(\text{Fitt}_0(I/J)) + E_U + (-1)^{r-1} e(J_{r-1}) & \text{if } r \text{ is even} \\ e(\text{Fitt}_0(I/J)) + E_U + (-1)^{r-2} e(J_{r-2}) + (-1)^{r-1} e(J_{r-1}) & \text{if } r \text{ is odd} \end{cases}$$

$$(5) \quad -\ell(R/I_r(\tilde{N})) = \begin{cases} -e(\text{Fitt}_0(I/J')) - E_N - e(J'_{r-1}) & \text{if } r \text{ is even} \\ -e(\text{Fitt}_0(I/J')) - E_N - (-1)^{r-2} e(J'_{r-2}) - (-1)^{r-1} e(J'_{r-1}) & \text{if } r \text{ is odd} \end{cases}$$

$$(6) \quad \ell(R/\text{Fitt}_0(I/I')) = e(\text{Fitt}_0(I/I')) + E_I.$$

Moreover by Theorem 1.2,

$$\begin{aligned} \ell(R/I_r(\tilde{N})) - \ell(R/\text{Fitt}_0(I/I')) &= \ell(I/J') - \ell(I/I') \\ &= (\ell(R/J') - \ell(R/I)) - (\ell(R/I') - \ell(R/I)) \\ &= e(J') - e(I') \\ &= e(J) - e(I). \end{aligned}$$

Thus, we have

$$(7) \quad \ell(R/I_r(\tilde{N})) - \ell(R/\text{Fitt}_0(I/I')) = e(J) - e(I).$$

Theorem 1.2 also shows

$$(8) \quad \text{br}(M) = \text{br}(U) = \ell(R/I_r(\tilde{U})).$$

Now, by adding equations (4), (5), (6), (7) and applying (8) we obtain the multiplicity formula in Theorem 3.3. \square

We state the rank two and rank three cases as a corollary. The multiplicity formulas have a more simple form in these cases.

Corollary 3.4. *We use the assumption of 3.2.*

(a) *If $r = 2$ then*

$$\text{br}(M) = e(J) - e(I) + e(\text{Fitt}_0(I/J)) - e(\text{Fitt}_0(I/J')).$$

(b) *If $r = 3$ then*

$$\text{br}(M) = e(J) - e(I) + e(\text{Fitt}_0(I/J)) - e(\text{Fitt}_0(I/J')) + e(\text{Fitt}_0(I/I')).$$

Proof. These results follow immediately from Theorem 3.3. If $r = 2$, then the ideal I is its own minimal reduction and $e(\text{Fitt}_0(I/I')) = 0$. \square

Remark 3.5. It should be pointed out that if a minimal reduction J' of J is general enough, then there exist minimal reductions U of M such that Assumption 3.2 is satisfied. The following example shows that the formula of Corollary 3.4(a) fails for a specific J' , and therefore 3.2 does not hold for this J' .

Let $R = k[x, y]_{(x, y)}$ and M be a finitely generated module of finite colength in a free module F of rank 2 such that the presenting matrix of F/M is

$$\begin{pmatrix} x^{16} & 0 & 0 & x^5y^4 & -y^{14} \\ 0 & y^{10} & x^8y^4 & 0 & x^{20} \end{pmatrix}.$$

Then $F/M \cong I/J$ where $I = (x^{20}, y^{14})$ and $J = (x^{36}, x^{25}y^4, x^8y^{18}, y^{24})$. Note that $J' = (x^{36} + y^{24}, x^{25}y^4)$ is a minimal reduction of J . The value on the right-hand side of the formula of Corollary 3.4(a) is

$$e(J) - e(I) + e(\text{Fitt}_0(I/J)) - e(\text{Fitt}_0(I/J')) = 744 - 280 + 546 - 594 = 416$$

while $\text{br}(M) = 420$ (see [14, page 50] for details).

This example also shows that $e(\text{Fitt}_0(I/J'))$ is not independent of the choice of J' .

4. A GRAPHICAL INTERPRETATION OF THE BUCHSBAUM-RIM MULTIPLICITIES

In this section, we consider modules of rank two arising from monomial ideals. We compare our formulas to the result of E. Jones [12], who presented a graphical computation of the Buchsbaum-Rim multiplicity in this case.

We assume $R = k[x, y]_{(x, y)}$ where k is a field, and let \mathfrak{m} denote the maximal ideal of R . Let I and J be \mathfrak{m} -primary monomial ideals with $J \subset \mathfrak{m}I$, $\mu(I) = 2$ and $\mu(J) \leq 3$. Let F be a free module of rank 2 and M a submodule of F such that $M/F \cong I/J$. Jones computes the Buchsbaum-Rim multiplicity of M and shows that $\text{br}(M) = e(J) - e(I)$ with a few exceptions. For this one may assume that k is infinite.

We write $I = (x^s, y^t)$ and may assume that $J = (x^{s+i}, x^d y^{t+e}, y^{t+j})$. The module M can be taken to be the image in $F = R^2$ of the matrix

$$\widetilde{M} = \begin{pmatrix} -y^t & x^i & 0 & 0 \\ x^s & 0 & x^d y^e & y^j \end{pmatrix}$$

In [12] the modules M are classified into seven cases: In Figure *c*, the point $T(s, t)$ corresponds to the monomial $x^s y^t$ and similarly for other points including those in Figures *a*'s and *b*'s.

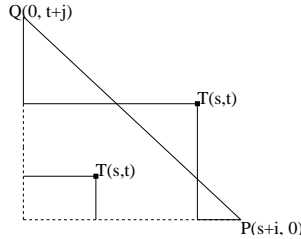


Figure *c*.

If T is above the line segment \overline{PQ} , then there are four cases determined by the relative positions of the point $B(d, t+e)$ and \overline{TQ} , \overline{PQ} , \overline{AQ} as shown in Figures a1–a4, where \overline{AQ} is parallel to \overline{PT} :

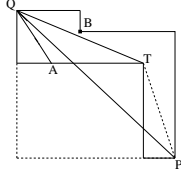


Figure a1.

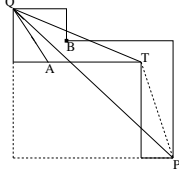


Figure a2.

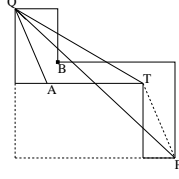


Figure a3.

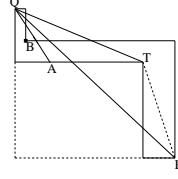


Figure a4.

If T in Figure c is below \overline{PQ} , there are three cases determined by the relative positions of B and \overline{PQ} , \overline{PT} as shown in Figures b1–b3:

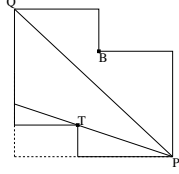


Figure b1.

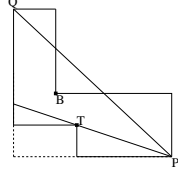


Figure b2.

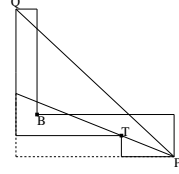


Figure b3.

For the cases in Figures a1 and b1, let U be the submodule of $F = R^2$ generated by the columns in the matrix

$$\tilde{U} = \begin{pmatrix} -y^t & x^i & 0 \\ x^s & 0 & y^j \end{pmatrix}.$$

Then U is a minimal reduction of the module M . Notice that the first column in \tilde{U} is the syzygy of the ideal I and the image of U in J is a minimal reduction J' of J . Therefore in 3.2, we may take \tilde{N} to be \tilde{U} and \tilde{L} to be \tilde{U} with the first column repeated. By performing row operations on \tilde{U} and by adding suitable linear combinations of columns of \tilde{U} to later columns we have all the conditions required for Corollary 3.4. Since J' is the image of U in J and U is a reduction of M , $\text{Fitt}_0(I/J')$ is a reduction of $\text{Fitt}_0(I/J)$. Hence by Corollary 3.4(a),

$$\text{br}(M) = e(J) - e(I).$$

This was also shown in in [12].

In Figures a4, b2 and b3, let U be the submodule of F generated by the columns in the matrix

$$\tilde{U} = \begin{pmatrix} -y^t & x^i & 0 \\ x^s & y^j & x^d y^e \end{pmatrix}.$$

By the same argument, $\text{br}(M) = e(J) - e(I)$.

For the remaining cases, the modules of Figures a2 and a3, we use the computation of the Buchsbaum-Rim multiplicity given in [12]. There it is shown that M is a reduction of the module generated by M and the vector $(0, x^s)$ in F , which is a direct sum of two monomial ideals. This allows for a computation of $\text{br}(M)$. Thus in the case of Figure a2,

$$(9) \quad \text{br}(M) = e(J) - e(I) - 2 \cdot \text{dark area},$$

where the dark area is the area of the triangle TBQ indicated in the following Figure $a2'$. On the other hand, the modules of Figure $a3$ have Buchsbaum-Rim multiplicity

$$(10) \quad \text{br}(M) = e(J) - e(I) - 2 \cdot \text{dark area} + 2 \cdot \text{light area},$$

where the dark area is the area of the triangle TBQ and the light area is the area of the triangle PBQ as indicated in Figure $a3'$.

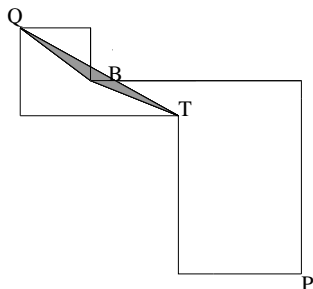


Figure $a2'$.

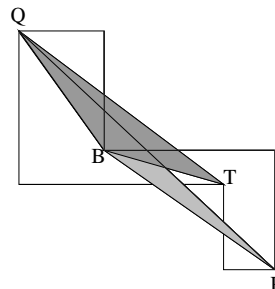


Figure $a3'$.

By Corollary 3.4(a), the extra terms subtracted in (9) and (10) are exactly

$$e(\text{Fitt}_0(I/J')) - e(\text{Fitt}_0(I/J))$$

for some general minimal reduction J' of J that satisfies 3.2. We remark that in the first five cases, since $\text{Fitt}_0(I/J)$ has a simple form, one can find a minimal reduction U of M that is close to being monomial. For the cases $a2$ and $a3$, this is much more complicated.

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